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Charges of exceptionally twisted branes

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Abstract: The charges of the exceptionally twisted (D4 with triality and E6 with charge conjugation) D-branes of WZW models are determined from the microscopic/CFT point of view. The branes are labelled by twisted representations of the affine algebra, and their charge is determined to be the ground state multiplicity of the twisted representation. It is explicitly shown using Lie theory that the charge groups of these twisted branes are the same as those of the untwisted ones, confirming the macroscopic K-theoretic calculation. A key ingredient in our proof is that, surprisingly, the G2 and F4 Weyl dimensions see the simple currents of A2 and D4, respectively.

Keywords: Conformal and W Symmetry, D-branes, Quantum Groups.

Contents

1. Introduction

Conserved charges of D-branes in String Theory, to a very large part, determine their effective dynamics. As such, determining these charges and the associated charge groups provides significant information regarding the D-branes. For strings propagating on a group manifold, i.e. a g_k-WZW model, these charges can be determined using the underlying CFT [1]. WZW models possess an extremely rich variety of D-brane dynamics directly attributable to the additional affine Lie structure, which is preserved by the D-branes.

In addition to the standard untwisted branes, WZW models also possess D-branes which preserve the affine symmetry only up to a twist, the so-called "twisted" branes. For every automorphism ω of the finite dimensional Lie algebra $\overline{g}$ of the affine Lie algebra g, there exist ω-twisted D-branes. It is sufficient to consider outer automorphisms only, and as such, only automorphisms determined by symmetries of the Dynkin diagram of $\overline{g}$ [2, 3]. Such twists exist for the A_n's, D_n's, and E_6, where ω in each case is an order two symmetry referred to as charge conjugation (or chirality flip in the case of D_n with n even), and for D_4, where ω is an order three symmetry referred to as triality. The microscopic analysis for twisted D-branes started with [4], and a study at large affine level was done in [5]. The charges and charge groups for the order-two twisted A_n and D_n D-branes have been calculated in [6] (up to some conjectures). This paper deals with the remaining cases of D_4 with triality and E_6 with charge conjugation.

The computations for D_4 and E_6 presented here, are purely Lie theoretic, and are done from a "microscopic"/CFT point of view. These calculations provide confirmations for the results for the charge group obtained "macroscopically"/geometrically using K-theory [7]. However, the K-theoretic arguments only determine the charge group and not the charges themselves, so the calculations done here provide significantly more information about the D-branes.

We also prove some Lie theoretic identities which warrant further study. The most surprising, and likely important, of these are that G_2 and F_4 see the simple currents of A_2 and D_4, respectively. More precisely, for arbitrary choice of level k, the simple currents J^i of A_2 permute the integral weights a' of G_2 in such a way that

$$\dim_{G_2}(J^i a') = \dim_{G_2}(a') \mod M_{G_2}\,, \tag{1.1}$$

where M_{G_2} is an integer given next section. Similarly, the 4 simple currents J of D_4 permute the integral weights of F_4 in such a way that

$$\dim_{F_4}(Jb') = \dim_{F_4}(b') \mod M_{F_4}\,, \tag{1.2}$$

where likewise M_{F_4} is given next section.

We first provide a brief summary of the description of untwisted D-brane charges in CFT, as well as the order-two twisted D-branes of A_n and D_n. Subsequently, we deal with the exceptional cases of D_4 and E_6. The non-trivial Lie theoretic identities, which are needed along the way, are stated and proved in the appendices.

2. Overview of WZW D-brane charges in CFT

The WZW models of relevance here are the ones on simply connected compact group manifolds (partition function given by charge conjugation). D-branes that preserve the full affine symmetry are labelled by the level k integrable highest weight representations $P_+^k(g)$ of the affine algebra g. They are solutions of the "gluing" condition

$$J(z) = \overline{J}(\overline{z})\,, \qquad z = \overline{z}\,, \tag{2.1}$$

where $J, \overline{J}$ are the chiral currents of the WZW model [8].

The charge q_μ of the D-brane labelled by μ satisfies

$$\dim(\lambda)\,\mathrm{q}_\mu = \sum_{\nu\in P_+^k(g)} N_{\lambda\mu}^{\nu}\mathrm{q}_\nu \mod M\,, \tag{2.2}$$

where $\lambda \in P_+^k(g)$, $N_{\lambda\mu}^{\nu}$ are the g_k-affine fusion rules, and $\dim(\lambda) = \dim(\overline{\lambda})$ denotes the dimension of the $\overline{g}$ representation whose highest weight is the finite part of the affine weight λ — in this paper we freely interchange the affine weight λ with its finite part $\overline{\lambda}$, which is unambiguous since the level will always be understood. For a finite level k, this relationship (2.2) is only true modulo some integer M, and the charge group of these D-branes is then $\mathbb{Z}/M\mathbb{Z}$, where M is the largest positive integer such that (2.2) holds. We are

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assuming here that the only common divisor of all the q_μ is 1 (if they do have a common divisor, then this factor can be divided out). Without loss of generality, we assume the normalization $\mathrm{q}_0 = 1$. If we take μ to be the trivial representation 0, then clearly

$$\mathrm{q}_\lambda = \dim(\lambda)\,. \tag{2.3}$$

The integer M is then the largest integer such that

$$\dim(\lambda)\dim(\mu) = \sum_{\nu\in P_+^k(g)} N_{\lambda\mu}^{\nu}\dim(\nu) \qquad \mathrm{mod}\, M \tag{2.4}$$

holds. It has been conjectured (and proved for the A_n and the C_n series) in [9, 10, 7] that the integer M is always of the form

$$M = \frac{k+h^\vee}{\gcd(k+h^\vee, L)}\,, \tag{2.5}$$

where $h^\vee$ is the dual Coxeter number of $\overline{g}$ and L is a k-independent integer given in table 1.[1]

Algebra	$h^\vee$	L
A_n	$n+1$	$\mathrm{lcm}(1,2,\dots,n)$
B_n	$2n-1$	$\mathrm{lcm}(1,2,\dots,2n-1)$
C_n	$n+1$	$2^{-1}\mathrm{lcm}(1,2,\dots,2n)$
D_n	$2n-2$	$\mathrm{lcm}(1,2,\dots,2n-3)$
E_6	12	$\mathrm{lcm}(1,2,\dots,11)$
E_7	18	$\mathrm{lcm}(1,2,\dots,17)$
E_8	30	$\mathrm{lcm}(1,2,\dots,29)$
F_4	9	$\mathrm{lcm}(1,2,\dots,11)$
G_2	4	$\mathrm{lcm}(1,2,\dots,5)$

Table 1: The dual Coxeter numbers and charge group integer L for the simple Lie algebras.

WZW models also possess D-branes that only preserve the affine symmetry up to some twist. For every automorphism of the finite dimensional algebra $\overline{g}$, ω-twisted D-branes can be constructed. These are solutions of the "gluing" condition

$$J(z) = \omega\cdot\overline{J}(\overline{z})\,, \qquad z=\overline{z}\,, \tag{2.6}$$

where $J, \overline{J}$ are the chiral currents of the WZW model. These D-branes are labelled by the ω-twisted highest weight representations of g_k. The charge group is of the form $\mathbb{Z}_{M^\omega}$, where M^ω is the twisted analogue of the integer M from the untwisted case. The charge carried by the D-brane labelled by the ω-twisted highest weight a has an integer charge q_a^ω, such that

$$\dim(\lambda)\,\mathrm{q}_a^\omega = \sum_b \mathcal{N}_{\lambda a}^{b}\,\mathrm{q}_b^\omega \quad \mathrm{mod}\, M^\omega\,, \tag{2.7}$$

where $\mathcal{N}_{\lambda a}^{b}$ are the NIM-rep coefficients that appear in the Cardy analysis of these D-branes. M^ω is the largest integer such that (2.7) holds, again assuming that all the charges q_a^ω are relatively prime integers. However the difficulty in carrying over the analysis from the untwisted case is that there is no brane label a playing the role of the identity field, and thus we need to resort to a slightly different, and more complicated, analysis to determine the charges and M^ω.

[1]It was suggested initially in [10] that there were exceptional values for M at low levels k. However, this issue was subsequently resolved in [7], where it was proved that there are no exceptional cases using K-theory. We will show that there are no exceptional cases at low levels in both algebras, on the CFT and Lie theory side, when we prove uniqueness of the solutions.

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It was suggested in [12, 11] that the NIM-rep coefficients $\mathcal{N}_{\lambda a}^{b}$ are actually the twisted fusion rules that describe the WZW fusion of the twisted representation a with the untwisted representation λ to give the twisted representation b. Thus the conformal highest weight spaces of all three representations λ, a, and b form representations of the invariant horizontal subalgebra $\overline{g}^{\omega}$ that consists of the ω-invariant elements of $\overline{g}$ (For details on such matters, see [9]). The twisted fusion rules are a level k truncation of the tensor product coefficients of the horizontal subalgebra. This establishes a parallel with the untwisted case, where the untwisted fusion rules are the level k truncation of the tensor product coefficients of $\overline{g}$. Thus by analogy with (2.3), we can make the ansatz

$$\mathrm{q}_a^{\omega} = \dim_{\overline{g}^{\omega}}(a)\,, \tag{2.8}$$

i.e. the charge is simply the $\overline{g}^{\omega}$-Weyl dimension of the finite part of the twisted weight a. Using this ansatz the integer M^{ω} was calculated in [6] for the chirality flip twisted A_n and D_n series, and it was also shown that, up to rescaling, (2.8) is the unique solution to (2.7).

The remaining non-trivial cases of triality twisted D_4 and charge conjugation twisted E_6 are dealt with in this paper, and require some nontrivial Lie theory, especially pertaining to twisted affine Lie algebras. The relevant background can be found in [11, 9, 13].

3. Triality twisted D_4 brane charges

D_4 has five non-trivial conjugations, whose NIM-reps can all be determined from analyzing just the ones corresponding to chirality flip (which has already be done in [11, 6]) and triality. The latter is an order three automorphism of the Dynkin diagram ω that sends the Dynkin labels $(\lambda_0; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ to $(\lambda_0; \lambda_4, \lambda_2, \lambda_1, \lambda_3)$. Thus the relevant twisted algebra here is $D_4^{(3)}$ with a horizontal subalgebra G_2, labelling ω-invariant states. Thus boundary states are labelled by triples $(a_0; a_1, a_2)$ where the level $k = a_0 + 2a_1 + 3a_2$. In [11], it is shown how to express the twisted NIM-reps in terms of A_2 fusion rules at level $k+3$ via the branching $D_4 \supset G_2 \supset A_2$:

$$\mathcal{N}_{\lambda a}^{b} = \sum_{i=0}^{2} \sum_{\gamma''} b_{\gamma''}^{\lambda} \left(N_{J^i\gamma'', a''}^{b''} - N_{J^i\gamma'', Ca''}^{b''} \right), \tag{3.1}$$

where C denotes charge conjugation in A_2, which takes a dominant A_2 weight to its dual by interchanging the finite Dynkin labels and J is the simple current of A_2 that acts by cyclic permutation of the Dynkin labels of the $A_2^{(1)}$ weights, and the $b_{\gamma''}^{\lambda}$ are the $D_4 \supset G_2 \supset A_2$ branching rules (see for example [14]). The relation between D_4 boundary states and the weights of $G_2^{(1)}$ and $A_2^{(1)}$ is given by the identifications of the appropriate Cartan subalgebras. Explicitly, we write [11]

$$a' = \iota(a_0; a_1, a_2) = (a_0 + a_1 + a_2 + 2; a_2, a_1) \in P_+^{k+2}(G_2)\,, \tag{3.2}$$

$$a'' = \iota' a' = \iota'\iota(a_0; a_1, a_2) = (a_0 + a_1 + a_2 + 2; a_2, a_1 + a_2 + 1) \in P_+^{k+3}(A_2)\,. \tag{3.3}$$

In the following, level k-D_4 quantities (weights and boundary states) are unprimed, while the corresponding level $k+2$-G_2 weights and level $k+3$-A_2 weights are singly and doubly primed, respectively.

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Following [6], we make the ansatz that the charge q_a^ω is the G_2 Weyl dimension of the horizontal projection (finite part) of the weight i.e.

$$\mathrm{q}_a^\omega = \dim_{G_2}(a')\,. \tag{3.4}$$

Then for an arbitrary dominant integral weight λ of D_4 the left hand side of (2.7) reads:

$$\begin{aligned}\dim_{D_4}(\lambda)\dim_{G_2}(a') &= \sum_{\gamma'} b^\lambda_{\gamma'}\dim_{G_2}(\gamma')\dim_{G_2}(a') \\ &= \sum_{\gamma'} b^\lambda_{\gamma'} \sum_{b'\in P_+^{k+2}(G_2)} N_{\gamma' a'}^{b'}\dim_{G_2}(b') \qquad \mathrm{mod}\, M_{G_2}\,,\end{aligned} \tag{3.5}$$

where $b^\lambda_{\gamma'}$ are the $D_4\supset G_2$ branching rules, and in the second line we have used (2.2) for the untwisted G_2 branes at level $k+2$. Now from table 1 we know that at level $k+2$ M_{G_2} is the same as M_{D_4} at level k:

$$M_{G_2} = M_{D_4} = \frac{k+6}{\gcd(k+6, 2^2.3.5)}\,, \tag{3.6}$$

and so (3.5) holds $\mathrm{mod}\, M_{D_4}$. Now G_2 fusion rules at level $k+2$ can be written in terms of the level $k+3$ fusion rules of A_2 following [15]

$$N_{\gamma' a'}^{b'} = \sum_{\gamma''} b^{\gamma'}_{\gamma''}\left[N_{\gamma'' a''}^{b''} - N_{\gamma'' C a''}^{b''}\right], \tag{3.7}$$

where $b^{\gamma'}_{\gamma''}$ are the $G_2\supset A_2$ branching rules. Using this and the fact that $\sum_{\gamma'} b^\lambda_{\gamma'} b^{\gamma'}_{\gamma''} = b^\lambda_{\gamma''}$, we rewrite the left hand side of (2.7) as

$$\text{L.H.S.} = \sum_{\gamma''} b^\lambda_{\gamma''} \sum_{b'\in P_+^{k+2}(G_2)} \left[N_{\gamma'' a''}^{b''} - N_{\gamma'' C a''}^{b''}\right]\dim_{G_2}(b') \qquad \mathrm{mod}\, M_{D_4}\,. \tag{3.8}$$

In order to relate this to the right hand side of (2.7) where the summation is only over the boundary states of triality twisted D_4, we need to restrict the summation (3.8) somehow to the set $\mathcal{D} = Im(\iota'\iota)$ of images $b\mapsto b''$ under (3.2), (3.3). To do this, we first describe the relevant sets precisely.

An $A_2^{(1)}$ weight $(b_0'';b_1'',b_2'')$ belongs to $\mathcal{D}$, $J\mathcal{D}$, or $J^2\mathcal{D}$ respectively, if

$$\begin{aligned}\mathcal{D}:&\;\; b_0'' > b_2'' > b_1'' \geq 0\,,\\ J\mathcal{D}:&\;\; b_1'' > b_0'' > b_2'' \geq 0\,,\\ J^2\mathcal{D}:&\;\; b_2'' > b_1'' > b_0'' \geq 0\,,\end{aligned} \tag{3.9}$$

where J is the A_2 simple current acting on $A_2^{(1)}$ weights by $J(a_0'';a_1'',a_2'') = (a_2'';a_0'',a_1'')$.

The set $\mathcal{G} = \iota'(P_+^{k+2}(G_2))$ of images of (3.3) (the set over which we are summing in (3.8)) only has the constraint $b_2'' > b_1'' \geq 0$. Thus a moment of thought will show that

$$\mathcal{G} = \mathcal{D}\cup J^2\mathcal{D}\cup CJ\mathcal{D}\cup\mathcal{B}\,, \tag{3.10}$$

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where $\mathcal{B}$ consists of weights in $\mathcal{G}$ such that either $b''_0 = b''_1$ or $b''_0 = b''_2$. The following hidden symmetries are established in the appendices:

$$b' \in P^{k+2}(G_2) \Rightarrow \dim_{G_2}(Jb') = \dim_{G_2}(J^2 b') = \dim_{G_2}(b') \mod M_{D_4}\,, \tag{3.11}$$
$$b'' \in \mathcal{B} \Rightarrow \dim_{G_2}(b') = 0 \mod M_{D_4}\,, \tag{3.12}$$
$$b' \in P^{k+2}(G_2) \Rightarrow \dim_{G_2}(Cb') = -\dim_{G_2}(b')\,, \tag{3.13}$$

where C and J act on $G_2^{(1)}$ weights through conjugation by ι':

$$C(b'_0; b'_1, b'_2) = (b'_0; b'_1 + b'_2 + 1, -b'_2 - 2)\,, \tag{3.14}$$
$$J(b'_0; b'_1, b'_2) = (b'_1 + b'_2 + 1; b'_0, b'_1 - b'_0 - 1)\,. \tag{3.15}$$

Here and elsewhere, we write '$\dim_{G_2}(a')$' even when a' is not dominant, by formally evaluating the Weyl dimension formula for G_2 at a'. The minus sign in (3.13) indicates that Cb' won't be a dominant G_2 weight when b' is — indeed, C belongs to the Weyl Group of G_2.

Using these, we can rewrite (3.8) as

$$\begin{aligned}\text{L.H.S.} = \sum_{\gamma''} b^{\lambda}_{\gamma''} \Bigg[&\sum_{b''\in\mathcal{D}} [N^{b''}_{\gamma'' a''} - N^{b''}_{\gamma'' Ca''}] \dim_{G_2}(b') + \sum_{b''\in J^2\mathcal{D}} [N^{b''}_{\gamma'' a''} - N^{b''}_{\gamma'' Ca''}] \dim_{G_2}(b') + \\ &+ \sum_{b''\in J\mathcal{D}} [N^{Cb''}_{\gamma'' a''} - N^{Cb''}_{\gamma'' Ca''}] \dim_{G_2}(Cb') \Bigg] \mod M_{D_4}\,,\end{aligned} \tag{3.16}$$

where we note that there is no contribution from $\mathcal{B}$ due to (3.12). Using (3.13), the symmetry $N^{\nu''}_{\gamma'' a''} = N^{C\nu''}_{C\gamma'' Ca''}$, and the fact that the $D_4 \supset A_2$ branching has $b^{\lambda}_{\gamma''} = b^{\lambda}_{C\gamma''}$ for all γ'', to simplify the third sum in (3.16), we finally obtain

$$\text{L.H.S.} = \sum_{i=0}^{2} \sum_{\gamma''} b^{\lambda}_{\gamma''} \sum_{b''\in J^i\mathcal{D}} [N^{b''}_{\gamma'' a''} - N^{b''}_{\gamma'' Ca''}] \dim_{G_2}(b') \mod M_{D_4}\,. \tag{3.17}$$

But applying (3.1), (3.11), and the symmetries of fusion rules under the action of simple currents, we see that (3.17) also equals the right hand side of (2.7). Thus (2.7) is indeed satisfied by our ansatz

$$\mathrm{q}^{\omega}_a = \dim_{G_2}(a') \qquad \text{and} \qquad M^{\omega} = M_{D_4}\,, \tag{3.18}$$

that is, the charges are given by the Weyl dimension of the representation of the horizontal subalgebra, and the charge group is the same as in the untwisted case. As we show in section 5, the charges are unique up to a rescaling by a constant factor.

4. Charge conjugation twisted E_6 brane charges

E_6 has a non-trivial order two symmetry of the Dynkin diagram that sends the Dynkin labels $(\lambda_0; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ to $(\lambda_0; \lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_6)$. The relevant twisted algebra here is $E_6^{(2)}$, with a horizontal subalgebra F_4 labelling the ω invariant states. The boundary states are labelled by quintuples $(a_0; a_1, a_2, a_3, a_4)$ such that $k = a_0 + 2a_1 + 3a_2 + 4a_3 + 2a_4$. Again, in [11] it is shown how to express the twisted NIM-reps in terms of the untwisted fusion rules of D_4 at level $k+6$ via the branching $E_6 \supset F_4 \supset D_4$:

$$\mathcal{N}_{\lambda a}^{b} = \sum_{J} \sum_{\pi} \sum_{\gamma''} \epsilon(\pi)\, b^{\lambda}_{\gamma''}\, N^{b''}_{J\gamma'',\pi a''}\,, \tag{4.1}$$

where the sum over π is over all 6 conjugations of D_4 consisting of permutations of the 1st, 3rd, and 4th Dynkin labels, $b^{\lambda}_{\gamma''}$ are the $E_6 \supset F_4 \supset D_4$ branching rules and $\epsilon(\pi)$ is the parity of the permutation. The summation labelled by J is over the four simple currents of D_4: the identity, $J_\nu b'' = (b''_1; b''_0, b''_2, b''_4, b''_3)$, $J_s b'' = (b''_4; b''_3, b''_2, b''_1, b''_0)$, and $J_\nu J_s$. Note that each of these simple currents has order 2. The E_6 boundary states are related to $F_4^{(1)}$ and $D_4^{(1)}$ weights through the maps [11]

$$\begin{aligned} a' &= \iota(a_0; a_1, a_2, a_3, a_4) = (a_0 + a_1 + a_2 + a_3 + 3; a_4, a_3, a_2, a_1) \in P_+^{k+3}(F_4) &(4.2)\\ a'' &= \iota' a' = \iota' \iota(a_0; a_1, a_2, a_3, a_4) &(4.3)\\ &= (a_0 + a_1 + a_2 + a_3 + 3; a_1 + a_2 + a_3 + 2, a_4, a_3, a_2 + a_3 + 1) \in P_+^{k+6}(D_4)\,. \end{aligned}$$

These again correspond to the identification of the respective Cartan subalgebras. Unprimed quantities refer to level k-E_6 quantities, while their corresponding level $k+3$-F_4 and level $k+6$-D_4 weights are singly and doubly primed, respectively. Again, as explained in [11] and [15], these relations are established by examining the twisted version of the Verlinde formula.

Following [6] again, we take the ansatz that the charge q^{ω}_{a} is the F_4 Weyl dimension of the finite part of the weight i.e.

$$\mathrm{q}^{\omega}_{a} = \dim_{F_4}(a')\,. \tag{4.4}$$

Then for an arbitrary dominant integral weight λ of E_6 the left hand side of (2.7) reads:

$$\begin{aligned} \dim_{E_6}(\lambda)\dim_{F_4}(a') &= \sum_{\gamma'} b^{\lambda}_{\gamma'} \dim_{F_4}(\gamma')\dim_{F_4}(a') \\ &= \sum_{\gamma'} b^{\lambda}_{\gamma'} \sum_{b' \in P_+^{k+3}(F_4)} N^{b'}_{\gamma' a'} \dim_{F_4}(b') \qquad \bmod M_{F_4}\,, \end{aligned} \tag{4.5}$$

where $b^{\lambda}_{\gamma'}$ are the $E_6 \supset F_4$ branching rules, and in the second line we have used (2.2) for the untwisted F_4 branes at level $k+3$. From table 1 we know that at level $k+3$ M_{F_4} is the same as M_{E_6} at level k:

$$M_{F_4} = M_{E_6} = \frac{k+12}{\gcd(k+12, 2^3.3^2.5.7.11)}\,, \tag{4.6}$$

and so (4.5) also holds $\bmod M_{E_6}$.

Now F_4 fusion rules at level $k+3$ can be written in terms of the level $k+6$ fusion rules of D_4 following [15]

$$N^{b'}_{\gamma' a'} = \sum_{\gamma''} \sum_{\pi} \epsilon(\pi)\, b^{\gamma'}_{\gamma''}\, N^{b''}_{\gamma'' \pi a''}\,, \tag{4.7}$$

where $b^{\gamma'}_{\gamma''}$ are the $F_4 \supset D_4$ branching rules, and the $\pi \in S_3$ are as before. As explained in [11] and [15], this is obtained using the Verlinde formula by analyzing the subset of images of dominant integral weights under the branching. Now using this and the fact that $\sum_{\gamma'} b^{\lambda}_{\gamma'} b^{\gamma'}_{\gamma''} = b^{\lambda}_{\gamma''}$, we rewrite the left hand side of (2.7) as

$$\text{L.H.S.} = \sum_{\gamma''} \sum_{b' \in P^{k+3}_+(F_4)} \sum_{\pi} \epsilon(\pi)\, b^{\lambda}_{\gamma''}\, N^{b''}_{\gamma''\pi a''} \dim_{F_4}(b') \mod M_{E_6} \,. \tag{4.8}$$

In order to relate this to the right hand side of (2.7) where the summation is only over the boundary states of twisted E_6, we need to restrict the above summation somehow to the set $\mathcal{E} = Im(\iota'\iota)$ of images $b \mapsto b''$. To do this, we first describe the relevant sets precisely.

A $D_4^{(1)}$ weight $(b''_0; b''_1, b''_2, b''_3, b''_4)$ belongs to $\mathcal{E}, J_\nu\mathcal{E}, J_s\mathcal{E}$, or $J_\nu J_s\mathcal{E}$, where the J are the D_4 simple currents, if

$$\begin{aligned} \mathcal{E}: &\ b''_0 > b''_1 > b''_4 > b''_3 \geq 0\,, \\ J_\nu\mathcal{E}: &\ b''_1 > b''_0 > b''_3 > b''_4 \geq 0\,, \\ J_s\mathcal{E}: &\ b''_4 > b''_3 > b''_0 > b''_1 \geq 0\,, \\ J_\nu J_s\mathcal{E}: &\ b''_3 > b''_4 > b''_1 > b''_0 \geq 0\,. \end{aligned} \tag{4.9}$$

The set $\mathcal{F}$ of images of $P^{k+3}_+(F_4)$ under the $F_4 \supset D_4$ branching (the set over which we are summing in (4.8)) only has the constraints $b''_1 > b''_4 > b''_3 \geq 0$. Thus a moment of thought will show that

$$\mathcal{F} = \mathcal{E} \cup \pi_{143} J_\nu\mathcal{E} \cup \pi_{341} J_s\mathcal{E} \cup \pi_{413} J_\nu J_s\mathcal{E} \cup \mathcal{B}\,, \tag{4.10}$$

where π_{abc} is the D_4 conjugation taking the Dynkin labels 1,3 and 4 respectively to a, b and c, and $\mathcal{B}$ consists of weights in $\mathcal{F}$ such that either $b''_0 = b''_1$ or $b''_0 = b''_2$ or $b''_0 = b''_4$.

The following facts, where the π are the D_4 conjugations and the J are any of the D_4 simple currents, are proved in the appendices:

$$\dim_{F_4}(\pi b') = \epsilon(\pi) \dim_{F_4}(b') \quad \forall b' \in P^{k+3}(F_4)\,, \tag{4.11}$$

$$\dim_{F_4}(J b') = \dim_{F_4}(b') \mod M_{F_4} \quad \forall b' \in P^{k+3}(F_4)\,, \tag{4.12}$$

$$\dim_{F_4}(b') = 0 \mod M_{E_6} \quad \forall b'' \in \mathcal{B}\,. \tag{4.13}$$

The action of the D_4 conjugations π and simple currents J on $F_4^{(1)}$ weights can be easily obtained by converting $F_4^{(1)}$ weights to $D_4^{(1)}$ weights using ι', applying π or J, and then converting back to $F_4^{(1)}$ using $\iota'^{-1}(d''_0; d''_1, d''_2, d''_3, d''_4) = (d''_0; d''_2, d''_3, d''_4 - d''_3 - 1, d''_1 - d''_4 - 1)$. As for G_2, we write '$\dim_{F_4}(a')$' even when a' is not dominant, by formally applying the Weyl dimension formula. The factor $\epsilon(\pi)$ in (4.11) is the parity ± 1, and as before, each π belongs to the Weyl Group of F_4.

Using these, we can rewrite (4.8) as

$$\text{L.H.S.} = \sum_{\gamma''} b^{\lambda}_{\gamma''} \sum_{\pi} \epsilon(\pi) \Bigg[\sum_{b'' \in \mathcal{E}} N^{b''}_{\gamma''\pi a''} \dim_{F_4}(b') + \sum_{b'' \in J_\nu\mathcal{E}} N^{\pi_{143} b''}_{\gamma''\pi a''} \dim_{F_4}(\pi_{143} b') +$$

$$+\sum_{b''\in J_s\mathcal{E}} N_{\gamma''\pi a''}^{\pi_{341}b''}\dim_{F_4}(\pi_{341}b')+$$

$$\left.+\sum_{b''\in J_\nu J_s\mathcal{E}} N_{\gamma''\pi a''}^{\pi_{413}b''}\dim_{F_4}(\pi_{413}b')\right] \mod M_{E_6}\,, \tag{4.14}$$

where we note that there is no contribution from $\mathcal{B}$ due to (4.13). Exactly as for the argument last section, the symmetries of the fusion rules under π and simple currents, the symmetry $b^{\lambda}_{\pi\gamma''}=b^{\lambda}_{\gamma''}$ of the branching rules, together with the hidden symmetries (4.11), (4.12) and the expression (4.1), show that

$$\text{L.H.S.}=\sum_{\gamma''} b^{\lambda}_{\gamma''}\sum_J\sum_\pi \epsilon(\pi)\sum_{b''\in\mathcal{E}} N^{b''}_{J\gamma''\pi a''}\dim_{F_4}(b')=\text{R.H.S.}\mod M_{E_6}\,. \tag{4.15}$$

Thus, again, (2.7) is indeed satisfied by our ansatz

$$\mathrm{q}^{\omega}_a=\dim_{F_4}(a')\qquad\text{and}\qquad M^{\omega}=M_{E_6}\,, \tag{4.16}$$

that is, the charges are once again given by the Weyl dimension of the representation of the horizontal subalgebra, and the charge group is the same as in the untwisted case. As we show next, the charges are unique up to a rescaling by a constant factor.

5. Uniqueness

We need to show that the solutions found to the charge equation (2.7) in both the D_4 and E_6 cases are unique up to rescaling. To this end it is sufficient to prove that if the charge equation is satisfied by a set of integers $\tilde{q}_a$ modulo some integer $\tilde{M}$, then

$$\tilde{q}_a=\dim(a')\,\tilde{q}_0\mod\tilde{M}\,. \tag{5.1}$$

In this case, we can divide all charges by $\tilde{q}_0$, and the charge equation will still be satisfied if we also divide $\tilde{M}$ by $\gcd(\tilde{q}_0,\tilde{M})$. Finally, by an argument due to Fredenhagen [6] we get that $M':=\tilde{M}/\gcd(\tilde{q}_0,\tilde{M})$ must divide our M. Explicitly, by construction, M is the g.c.d. of the dimensions of the elements of the fusion ideal that quotients the representation ring in order to obtain the fusion ring. Since NIM-reps provide representations of the fusion ring, any element of the fusion ideal acts trivially i.e. $\dim(\alpha)\dim(a)=0\mod M'$ for any α in the fusion ideal. Thus, using the fact that the $\dim(a)$ are relatively prime integers, we see that M' must divide M. Thus any alternate solution $\tilde{q}_a,\tilde{M}$ to (2.7) which obeys (5.1), is just a rescaled version of our "standard" one q_a, M.

We will work with D_4, the proof for E_6 is similar and will be sketched at the end. The $G_2\subset D_4$ branching rules can be inverted: we can formally write

$$a'=\sum_\lambda \tilde{b}^a_\lambda\lambda\,, \tag{5.2}$$

where $\tilde{b}^a_\lambda$ are integers (possibly negative), $a'\in P_+(G_2)$, and the sum is over D_4 weights λ [15]. More precisely, (5.2) holds at the level of characters, where the domain of the D_4 ones is restricted to the ω-invariant vectors in the D_4 Cartan subalgebra, and the G_2 characters are evaluated at the image of those vectors by ι. To prove (5.2), it suffices to verify it for the G_2 fundamental weights, where we find $(1,0) = (0,1,0,0) - (1,0,0,0) + (0,0,0,0)$ and $(0,1) = (1,0,0,0) - (0,0,0,0)$. Since all other G_2 weights can be constructed from the fundamental ones by tensor products, every dominant G_2 weight can formally be inverted under the branching and written in terms of linear combinations of dominant integral D_4 weights. Then[2]

$$\begin{aligned}\dim_{G_2}(a')\,\tilde{q}_0 &= \sum_{\lambda} \tilde{b}^a_\lambda \dim_{D_4}(\lambda)\,\tilde{q}_0 \\ &= \sum_{\lambda} \tilde{b}^a_\lambda \sum_{b'} \mathcal{N}^b_{\lambda,0} \tilde{q}_b \mod \tilde{M}\,,\end{aligned} \tag{5.3}$$

where we have used the charge equation, which the $\tilde{q}_a$ satisfy modulo $\tilde{M}$ by assumption. Now we use the expression (3.1) to write (5.3) in terms of A_2 fusions: we get

$$\text{R.H.S.} = \sum_{\lambda} \tilde{b}^a_\lambda \sum_{b} \sum_{\gamma''} \sum_{j=0}^{2} b^\lambda_{\gamma''} \left[N^{b''}_{J^j\gamma'',0''} - N^{b''}_{J^j\gamma'',C0''} \right] \tilde{q}_b\,. \tag{5.4}$$

Now however, from (3.7) and [15], we can express this in terms of G_2 untwisted fusion rules, and using properties of the A_2 fusion rules under simple currents along the way, we obtain

$$\text{R.H.S.} = \sum_{\lambda} \tilde{b}^a_\lambda \sum_{b} \sum_{\gamma'} \sum_{j=0}^{2} b^\lambda_{\gamma'} N^{J^j b'}_{\gamma' 0'}\,\tilde{q}_b \mod \tilde{M}\,. \tag{5.5}$$

Note that $0'$ here is the G_2 vacuum (whereas $0''$, the image of $0'$ under ι', is not the A_2 vacuum), and thus

$$\text{R.H.S.} = \sum_{b} \sum_{j=0}^{2} \delta^{J^j b'}_{a'}\,\tilde{q}_b = \tilde{q}_a \mod \tilde{M}\,, \tag{5.6}$$

where we have used the facts that b' is never fixed by J and that the sets $\mathcal{D}$ and $J\mathcal{D}$ are disjoint(see (3.15)). Thus (5.1), and with it uniqueness, is established.

The proof for E_6 is virtually identical, except now we use the expression for untwisted F_4 fusion rules in terms of D_4 fusion rules found in [15].

6. Conclusion

In this paper, we have shown that the charge groups of the triality twisted D_4 and the charge conjugation twisted E_6 branes are identical to those of the untwisted D-branes. This

[2] In the following, we use the fact that both (2.7) and (3.1) (and (4.1) in the case of E_6) remain true for all dominant weights λ and not just affine ones. These expressions were obtained using ratios of S-matrices (see [11]), which can be interpreted as Lie algebra characters in the case of finite weights, and thus the NIM-reps $\mathcal{N}$ can be continued to include all dominant weights. This continuation also removes any subtleties in the comparison to K-theory by evaluating the charge constraint equation for all dominant weights λ. We thus can also see there are no exceptional situations at low levels.

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is in nice agreement with the K-theoretic calculation [7] and completes the exceptional cases not dealt with in [6]. Our calculations show that the charges of these twisted D-branes corresponding to the twisted representation a is the dimension of the highest weight space of the representation a. Thus from the string theoretic point of view, analogous to the situation with untwisted D-branes, the charge associated to the D-brane is the multiplicity of the ground state of the open string stretched between the fundamental $D0$-brane and the brane labelled by a in question. So, in the supersymmetric version of WZW models, the charge may be interpreted as an intersection index, motivating possible geometric interpretation of these results. The explicit computation of the charges is missing from the K-theoretic calculations, and has been supplied here.[3] There are no additional unproven conjectures made in this paper. All the arguments have been proved up to some conjectures needed from the untwisted cases (i.e. the content of table 1 for D_4, G_2, E_6, and F_4).

A number of non-trivial, and somewhat surprising Lie theoretic identities have been proved along the way. Some of the dimension formulae regarding the action of simple currents of a subalgebra on weights of the larger algebra indicate that there might exist interesting constraints on the larger algebra due to the underlying symmetry in the branchings. In some sense, the enlarged algebra "breaks" symmetries of the smaller algebra, but still "sees" the underlying symmetry (analogous to ideas of renormalization of quantum field theories with spontaneously broken symmetries.)

The Lie theoretic meaning of (3.13) and (4.11) is clear: the A_2 and D_4 conjugations C and $\pi \in S_3$ are elements of the Weyl groups of G_2 and F_4 respectively. The meaning of (3.11) and (4.12) is far less clear (though it has to do with the theory of equal rank subalgebras [17]), but it does suggest a far-reaching generalization whenever the Lie algebras share the same Cartan subalgebras — for example, $A_1 \oplus \cdots \oplus A_1$ (n copies) and C_n, or A_8 and E_8. Given any simple current J of any affine Lie algebra g at level k, it is already surprising that Weyl dimensions for the horizontal subalgebra $\overline{g}$ see the action of J via $\dim_{\overline{g}}(J\lambda) = \pm\dim_{\overline{g}}(\lambda) \mod M_{g_k}$. Far more surprising is that, at least sometimes, if two Lie algebras $\overline{g}$ and $\overline{g}'$ share the same Cartan subalgebras, then the Weyl dimensions of the first sees the simple currents J' of the second: $\dim_{\overline{g}}(J'\lambda) = \pm\dim_{\overline{g}}(\lambda) \mod M_{g_k}$.

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A. Dimension formulae and examples

We will use the following fact for proofs in both the D_4 and E_6 cases.

Fact 1. *Suppose K, L, N are arbitrary integers. Write $M = \frac{K}{\gcd(K,L)}$, and let $L = \prod_p p^{\lambda_p}$ and $N = \prod_p p^{\nu_p}$ be prime decompositions. Suppose we have integers f_i, d_i such that both*

[3] During the preparation of this manuscript the work of [16] has come to our attention, in which similar results are derived using different methodology.

$\prod_i f_i$ *and* $\prod_i (f_i - d_i K)$ *are divisible by* N. *Then*

$$\frac{\prod_i f_i}{N} = \frac{\prod_i (f_i - d_i K)}{N} \mod M\,, \tag{A.1}$$

provided that for each prime p *dividing* M *such that* $\lambda_p < \nu_p$, *it is possible to find* $0 \leq \alpha_{i,p} \leq \lambda_p$, *such that* $p^{\alpha_{i,p}}$ *divides* f_i *for each* i, *and* $\sum_i \alpha_{i,p} \geq \nu_p$.

The reason we can restrict to primes p dividing M is that p coprime to M are invertible modulo M, and so can be freely cancelled on both sides and ignored. For primes p dividing M, $a_i/p^{\alpha_{i,p}} = (a_i - \delta_i K)/p^{\alpha_{i,p}}$ holds $\mod M$, $\forall\, i$. If $\lambda_p \geq \nu_p$, choose $\alpha_{i,p}$ to be the exact power of p dividing a_i. The divisibility by N hypothesis will be automatically satisfied, because the products we will be interested in come from the Weyl dimension formula.

A.1 D_4 dimension formulae

For any integral weight $a'' = (a_0; a_1, a_2) \in P^{k+3}(A_2)$, we can substitute $\iota'^{-1}(a_0; a_1, a_2) = (a_0; a_1, a_2 - a_1 - 1) = a'$ into the Weyl dimension formula [18] of G_2, in order to express G_2-dimensions using A_2 Dynkin labels:

$$\dim_{G_2}(a') = \frac{1}{120}(a_2 - a_1)(a_2 + 1)(2a_2 + a_1 + 3)(a_2 + a_1 + 2)(a_2 + 2a_1 + 3)(a_1 + 1) \tag{A.2}$$

Theorem 1. $\forall\, a' \in P^{k+2}(G_2)$, $\dim_{G_2}(Ca') = -\dim_{G_2}(a')$.

This is an automatic consequence of the $a_1 \leftrightarrow a_2$ anti-symmetry of (A.2). In fact, C is in the Weyl group of G_2 and so more generally theorem 1 follows from the anti-symmetry of the Weyl dimension formula under Weyl group elements.

Theorem 2. $\dim_{G_2}(a') = \dim_{G_2}(Ja') = \dim_{G_2}(J^2 a') \mod M_{G_2} \quad \forall a' \in P^{k+2}(G_2)$.

Proof. Using (A.2) and ι', we get

$$\begin{aligned}\dim_{G_2}(J^2 a') = {} & \frac{1}{120}(a_1 + a_2 + 2)(a_1 + 2a_2 + 3 - K)(a_2 + 1 - K) \,* \\ & *(2a_1 + 3a_2 + 5 - K)(a_1 + 3a_2 + 4 - 2K)(a_1 + 1 + K)\,,\end{aligned}$$

where we put $K = k + 6$ and used the fact that $k = a_0 + 2a_1 + 3a_2$. In the notation of Fact 1, here $N = 120 = 2^3.3.5$, $L = 60 = 2^2.3.5$. From Fact 1, it suffices to consider the primes p with $\nu_p > \lambda_p$, i.e. $p = 2$. To show that $p = 2$ always satisfies the condition of Fact 1, i.e. that the $\alpha_{i,2}$ can be found for any choice of a_i, it suffices to verify it separately for the 16 possible values of $a_1, a_2 \mod 2^2$. Though perhaps too tedious to check by hand, a computer does it in no time. The proof for $\dim_{G_2}(Ja')$ is now automatic from theorem 1.

Theorem 3. *Given any* $b' \in P^{k+2}(G_2)$, *if* $Cb' = J^i b'$ *for some* i, *then* $\dim_{G_2}(b') = 0 \mod M_{G_2}$.

Proof. Write $b'' = (b_0; b_1, b_2) \in P^{k+3}(A_2)$. By theorem 1, it suffices to consider the case where $b_0 = b_2$. In this case we can write $k+3 = b_1 + 2b_2$. Thus, again using (A.2) we have

$$\dim_{G_2}(b') = \frac{1}{120}(b_1+1)(b_2+1)(b_1+b_2+2)(3b_2+3-K)K(3b_2+3-2K)\,.$$

The proof now proceeds as in theorem 2.

Of course given any weight $b'' \in \mathcal{B}$, $b' = \iota'^{-1}(b'')$ will obey the hypothesis of theorem 3, and so (3.12) follows. Note that combining theorems 1 and 2, we get that any weight b' as in theorem 3 will obey $\dim_{G_2}(b') = -\dim_{G_2}(b') \mod M_{G_2}$. Thus theorems 1 and 2 are almost enough to directly get theorem 3 (and in fact imply it for all primes $p \neq 2$).

A.2 E_6 dimension formulae

As before, use ι' and the Weyl dimension formula for F_4 to write the (formal) Weyl dimension of an arbitrary F_4 integral weight $b' \in P(F_4)$ in terms of the Dynkin labels of the D_4 weight $b'' = \iota'(b') = (b''_0; b''_1, b''_2, b''_3, b''_4)$. For convenience write $a_i = b''_i + 1$. Then we obtain

$$\begin{aligned}
&\frac{1}{2^{15}3^7 5^4 7^2 11} a_1 a_2 a_3 a_4 (a_1+a_2)(a_1+a_3)(a_1+a_4)(a_2+a_3)(a_2+a_4)(a_3+a_4) *\\
&*(a_1-a_3)(a_1-a_4)(a_4-a_3)(a_1+a_2+a_3)(a_1+a_2+a_4)(a_2+a_3+a_4) *\\
&*(a_1+a_2+a_3+a_4)(a_1+2a_2+a_3)(a_1+2a_2+a_4)(2a_2+a_3+a_4) * \qquad \text{(A.3)}\\
&*(a_1+2a_2+a_3+a_4)(2a_1+2a_2+a_3+a_4)(a_1+2a_2+2a_3+a_4)(a_1+2a_2+a_3+2a_4)\,.
\end{aligned}$$

Theorem 4. *For any F_4 weight $b' \in P(F_4)$ and any outer automorphism $\pi \in S_3$ of D_4,*

$$\dim_{F_4}(\pi b') = \epsilon(\pi)\dim_{F_4}(b')\,. \qquad \text{(A.4)}$$

This follows easily from the Weyl dimension formula (A.3) by explicitly using the action of the π on the weights. As with theorem 1, it expresses the anti-symmetry of Weyl dimensions under the Weyl group.

Theorem 5. *For all $F_4^{(1)}$ weights $b' \in P^{k+3}(F_4)$,*

$$\dim_{F_4}(b') = \dim_{F_4}(J_\nu b') = \dim_{F_4}(J_s b') = \dim_{F_4}(J_\nu J_s b') \mod M_{F_4}\,. \qquad \text{(A.5)}$$

From theorem 4, it suffices to consider only J_ν. The proof uses Fact 1 and an easy computer check for primes $p = 2, 3, 5, 7$, as in the proof of theorem 2.

Theorem 6. *Let $b' \in P^{k+3}(F_4)$ be any $F_4^{(1)}$ weight satisfying $\pi b' = Jb'$, for any D_4 simple current J and any order-2 outer automorphism π of D_4. Then $\dim_{F_4}(b') = 0 \mod M_{F_4}$.*

Proof. The proof of this follows automatically from theorem 4. Analogous to the situation for G_2, one of the factors in the dimension formula turns out to be K. So, accommodating the denominators as in Fact 1 will yield the term $\frac{K}{p^\alpha}$ for some $0 \le \alpha \le \lambda_p$. But this is 0 modulo M_{F_4}, for every prime p dividing M_{F_4}. Q.E.D.

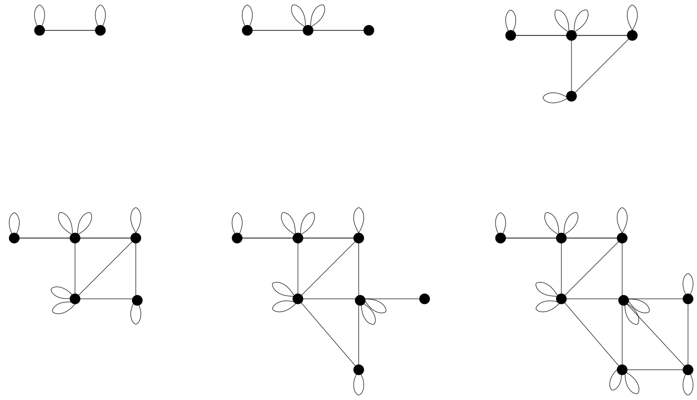

Figure 1: NIM-Reps for D_4 with Triality $k = 2, \dots, 7$.

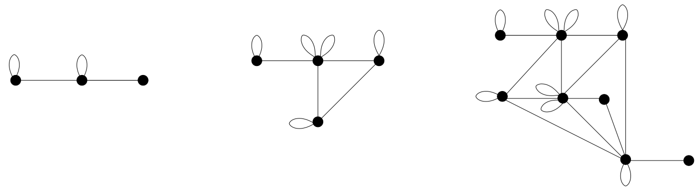

Figure 2: NIM-Reps for E_6 with Charge Conjugation $k = 2, 3, 4$.

A.3 NIM-reps and graphs

In this appendix we give explicit descriptions of some of the NIM-reps at low level for both D_4 and E_6 [11]. The NIM-rep graphs characterizing the matrix associated to the field $\lambda = \Lambda_1$ are given. The corresponding graph has vertices labeled by the rows (or columns) of $\mathcal{N}_\lambda$, and the vertex associated to i and j are linked by $(\mathcal{N}_\lambda)_{ij}$ lines.

A.4 A sample calculation

A suitable example to illustrate the situations considered is at level $k = 5$ for triality twisted D_4. In this case from table 1, we get $M_{D_4} = 11$. The boundary states are labelled by triples $(a_0; a_1, a_2)$ such that $k = a_0 + 2a_1 + 3a_2$. The boundary weights then are

$$[5,0,0], [3,1,0], [2,0,1], [1,2,0], [0,1,1]\,, \tag{A.6}$$

whose G_2 Weyl dimensions are respectively, 1, 7, 14, 27, 64.

The relevant NIM-rep graph is illustrated in figure 1. The charge equations (2.7) with $\lambda = \Lambda_1$ (fundamental representation of D_4 with dimension 8) thus are

$$\begin{aligned} 8\mathrm{q}_0 &= \mathrm{q}_0 + \mathrm{q}_1 \\ 8\mathrm{q}_1 &= \mathrm{q}_0 + 2\mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3 \\ 8\mathrm{q}_2 &= \mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3 + \mathrm{q}_4 \\ 8\mathrm{q}_3 &= \mathrm{q}_1 + \mathrm{q}_2 + 2\mathrm{q}_3 + \mathrm{q}_4 \\ 8\mathrm{q}_4 &= \mathrm{q}_2 + \mathrm{q}_3 + \mathrm{q}_4\,. \end{aligned} \tag{A.7}$$

The first three equations are identically true with $\mathrm{q}_0 = 1, \mathrm{q}_1 = 7, \mathrm{q}_2 = 14, \mathrm{q}_3 = 27$, and $\mathrm{q}_4 = 64$, and the last two equations are satisfied modulo $M_{D_4} = 11$.

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